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ON THE PROBLEM OF LINEAR ELASTICITY FOR AN INFINITE REGION CONTAINING A FINITE NUMBER OF NON-INTERSECTING SPHERICAL INHOMOGENEITIES

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Abstract—The problem of linear elasticity for an infinite region containing a finite number of non-intersecting spherical and, more generally, ellipsoidal inhomogeneities is attacked. The approach taken does not misrepresent geometry of inhomogeneities, although the continuity conditions at the interfaces are only approximately satisfied. The principal idea of the approach is to extend the method of Kachanov (1985, *Int. J. Fracture* **28**, R11–R19; 1987, *Int. J. Solids Structures* **23**, 23–43) for interacting cracks to the realm of the Eshelby equivalent inclusion method. The application to a test problem for two spherical cavities suggests that the approach can be useful for predictions of the overall response of composite materials and interfacial stress concentrations.

1. INTRODUCTION

In this paper we address the linear elasticity problem for an infinite region containing a finite number of non-intersecting spherical inhomogeneities. While a general closed-form solution to this problem is not feasible it is important to develop both accurate and efficient methods of approximate analysis. It is apparent that straightforward applications of numerical methods to this problem are extremely limited—computations become prohibitively expensive if the number of inhomogeneities exceeds two. Although a trivial approximation (the dilute solution), which neglects interactions among inhomogeneities, is attractive in terms of computations it may not be always sufficiently accurate. In the literature, there has been proposed a number of ways to interpret interactions among an infinite number of inhomogeneities (Christinsen, 1979; Hashin, 1983; Willis, 1983). It is, however, fair to state that the majority of those approaches homogenize the complex geometry of the problem and therefore can only be employed for estimates of the overall response. The other deficiency of homogenization procedures is the difficulty in formulating an adequate test problem. This work presents a simple and relatively accurate method of analysis which does not compromise the original geometry of the problem, although the continuity conditions at the interfaces are only approximately satisfied.

The number of publications dedicated to the problem of linear elasticity for N spherical inhomogeneities is rather small. The first analysis in this area is given by Sternberg and Sadowsky (1952) for the axisymmetric problem for two voids. Chen and Acrivos (1978a) provide a comprehensive treatment of the problem for two spherical inhomogeneities. Their analysis contains accurate results for moderately separated inhomogeneities and also exposes computational difficulties as interactions become stronger. One of the most important conclusions following from their work is that stress concentrations, induced by a neighbor, become important only when the inhomogeneities are very close to each other. For example, for two voids of radius a with the centers separated by $d = 2.1a$ (Fig. 1), stress concentrations under remote equal triaxial tension are only two times larger than those obtained for a single void. The data of Chen and Acrivos, as our finite element calculations suggest, are reliable up to $d = 2.25a$. The authors themselves give a conservative estimate $d = 3a$.

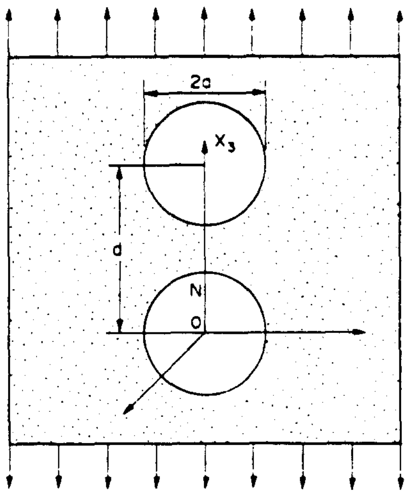

Fig. 1. An infinite region containing two spherical voids.

A more general approach to the problem for N inhomogeneities is suggested by Moschovidis and Mura (1975). They construct an approximate solution within the context of the formalism of Eshelby (1957, 1959, 1961). The key assumption, made by Moschovidis and Mura, is that the transformation strain within each domain is a polynomial in Cartesian coordinates. In order to completely reduce the boundary-value problem to a system of linear algebraic equations, the field around each inhomogeneity is represented by a Taylor's polynomial. Unfortunately, Moschovidis and Mura do not address the issue of convergence of their procedure and the selection of numerical examples is not representative of the problem.

A very simple estimate of weakly-interacting inhomogeneities is given by Willis and Acton (1976). This estimate is developed in conjunction with the so-called order c^2 calculations of the overall response pioneered by Batchelor and Green (1972) (c is the volume fraction of inhomogeneities). The approximation of Willis and Acton is correct to order $(a/d)^3$ and it predicts accurate results for well-separated inhomogeneities ($d \geqslant 2.5a$).

In our approach we decompose the original problem for N inhomogeneities into N disconnected problems for a single inhomogeneity with the consequent solution of each problem. This decomposition is motivated by an assumption made by Kachanov (1985, 1987) in a study of interacting cracks. We extend that assumption to spherical and, more generally, ellipsoidal inhomogeneities. Typically, formulation of the N problems requires inversion of a $6N \times 6N$ matrix. Each of these problems is stated for a single inhomogeneity which perturbs a complicated remote field. Solution to this class of problems can be obtained either by expanding this remote field into a Taylor's series (Moschovidis and Mura, 1975) or by using a finite element method.

In the next section we briefly review the Eshelby solution to the problems of a single inclusion and an inhomogeneity. The purpose of this review is to provide the background for the main problem both in terms of the concepts and notation. The third section discusses some mathematical aspects of the Eshelby solution which are directly related to the problem for N inhomogeneities. The analysis of the main problem is presented and tested in Sections 4 and 5, respectively. As a test we choose the problem for two equal voids subjected to axisymmetric remote stress. The discussion focuses on the extension of the method to the general case of ellipsoidal inhomogeneities, and its applications to the mechanics of composite materials.

We take the liberty of using both direct and index notation for tensors. Boldface lowercase Greek letters designate tensors of rank two, capital Latin letters tensors of rank four. Subscripts denote Cartesian components. Superscript n attributes the corresponding entry to the nth inhomogeneity. The finite element computations reported herein are performed with the program ABAQUS installed on a CRAY XMP computer.

2. SUMMARY OF THE ESHELBY SOLUTION

Let us consider an infinite region of a homogeneous linear elastic isotropic material with shear modulus μ and Poisson's ratio ν. There is an inclusion inside this region, a sphere $\omega(|\mathbf{x}| < a)$ which experiences uniform transformation strain $\boldsymbol{\beta}$. The open exterior of the inclusion, Ω, is termed matrix. The strain field, induced by the inclusion, can be formally written as

$$\boldsymbol{\varepsilon}(\mathbf{x}) = \begin{cases} \mathbf{S}\boldsymbol{\beta} & \text{if} \quad \mathbf{x} \in \omega \\ \mathbf{D}\boldsymbol{\beta} & \text{if} \quad \mathbf{x} \in \Omega. \end{cases} \tag{1}$$

Tensor S is a constant (Eshelby, 1957) with components

$$S_{1111} = S_{2222} = S_{3333} = \frac{7-5\nu}{15(1-\nu)}$$

$$S_{1122} = S_{2233} = S_{3311} = S_{1133} = S_{2211} = S_{3322} = \frac{5\nu-1}{15(1-\nu)}$$

$$S_{1212} = S_{2121} = S_{1313} = S_{3131} = S_{2323} = S_{3232} = \frac{4-5\nu}{15(1-\nu)}.$$

The remaining components of S are equal to zero. Tensor D is expressed in terms of potentials ϕ and ψ (Eshelby, 1959)

$$D_{ijkl}(\mathbf{x}) = \frac{1}{8\pi(1-\nu)}\{\psi_{,klij} - 2\nu\delta_{kl}\phi_{,ij} - (1-\nu)[\phi_{,kj}\delta_{il} + \phi_{,ki}\delta_{jl} + \phi_{,lj}\delta_{ik} + \phi_{,li}\delta_{jk}]\}, \tag{2}$$

with

$$\phi = \frac{4\pi a^3}{3}\frac{1}{|\mathbf{x}|} \quad \text{and} \quad \psi = \frac{4\pi a^3}{3}|\mathbf{x}| + \frac{4\pi a^5}{15}\frac{1}{|\mathbf{x}|}. \tag{3}$$

Formula (2), although with different expressions for the potentials, can be used for tensor S as well. Equation (5) of the next section is a general form of (1)–(3) valid for both S and D.

The inclusion problem can be employed as the basis in constructing the solution for an infinite body containing a spherical inhomogeneity. In this case the subdomain ω is occupied by a different linear elastic material. The stiffness tensors of the matrix and inhomogeneity materials are C and C*, respectively. The matrix is subjected to a remote uniform strain $\boldsymbol{\varepsilon}^{\infty}$. The perturbation, $\boldsymbol{\varepsilon}$, of the remote field can be simulated using an inclusion with the appropriately chosen equivalent transformation strain, $\boldsymbol{\beta}$. It is straightforward to verify that the necessary and sufficient condition for the equivalence between the fields induced by the inhomogeneity and the inclusion is

$$\mathbf{C}^*(\boldsymbol{\varepsilon}^{\infty} + \boldsymbol{\varepsilon}) = \mathbf{C}(\boldsymbol{\varepsilon}^{\infty} + \boldsymbol{\varepsilon} - \boldsymbol{\beta}) \quad \text{in} \quad \omega. \tag{4}$$

This condition and the relation $\boldsymbol{\varepsilon} = \mathbf{S}\boldsymbol{\beta}$ in ω form a closed system of algebraic equations for the equivalent transformation strain. An explicit expression for $\boldsymbol{\beta}$ can be found in Mura (1982, p. 156). After $\boldsymbol{\beta}$ has been determined the strain perturbation in the matrix is calculated from eqns (1)–(3).

Eshelby (1961) further derives a more general result. He demonstrates that if $\boldsymbol{\beta}$ is a polynomial of degree n in Cartesian coordinates, then the strain inside the inclusion is a sum of monomials of degree n, $n-2$, $n-4$, etc. Although in this case the mathematical expressions are somewhat lengthy, the structure of the approach is essentially the same as for a uniform transformation strain. This generalization for the inclusion problem immediately suggests that the inhomogeneity problem can also be extended to the general case of a polynomial remote strain. A very clear presentation of these topics is given by Mura (1982).

3. PROPERTIES OF THE ESHELBY SOLUTION

In this section we will derive certain important properties of tensors $\mathbf{S}$ and $\mathbf{D}$ in order to provide a consistent basis for the analysis of the problem for N inhomogeneities.

Property 1. *If* $\boldsymbol{\varepsilon}$ *is the strain inside the inclusion* ω, *induced by a transformation strain* β, *then*

$$\langle \boldsymbol{\varepsilon} \rangle = \mathbf{S} \langle \boldsymbol{\beta} \rangle,$$

where the brackets denote the volume average in ω.

In order to prove this relation we use an integral representation of (1) (Mura, 1982, p. 33):

$$\varepsilon_{ij}(\mathbf{x}) = -\frac{1}{2}\int_{\omega} C_{klmn}\beta_{mn}(\mathbf{x}')\,[G_{ik,lj}(\mathbf{x},\mathbf{x}') + G_{jk,li}(\mathbf{x},\mathbf{x}')]\,\mathrm{d}\mathbf{x}', \tag{5}$$

where $G_{ij}(\mathbf{x},\mathbf{x}')$ is the fundamental solution for an infinite domain. Direct integration of (5) gives

$$\begin{aligned}
\langle \varepsilon_{ij} \rangle &= \frac{1}{V_{\omega}}\int_{\omega} \varepsilon_{ij}(\mathbf{x})\,\mathrm{d}\mathbf{x} \\
&= -\frac{1}{2V_{\omega}}\int_{\omega}\int_{\omega} C_{klmn}\beta_{mn}(\mathbf{x}')\,[G_{ik,lj}(\mathbf{x},\mathbf{x}') + G_{jk,li}(\mathbf{x},\mathbf{x}')]\,\mathrm{d}\mathbf{x}'\,\mathrm{d}\mathbf{x} \\
&= \left\{-\frac{1}{2}\int_{\omega} C_{klmn}[G_{ik,lj}(\mathbf{x}',\mathbf{x}) + G_{jk,li}(\mathbf{x}',\mathbf{x})]\,\mathrm{d}\mathbf{x}\right\}\left\{\frac{1}{V_{\omega}}\int_{\omega}\beta_{mn}(\mathbf{x}')\,\mathrm{d}\mathbf{x}'\right\} \\
&= S_{ijmn}\langle \beta_{mn} \rangle.
\end{aligned}$$

In this calculation V_{ω} denotes the volume of ω. We have used the symmetry of tensor $G_{ij,kl}(\mathbf{x}',\mathbf{x})$ in arguments $\mathbf{x}$ and $\mathbf{x}'$, and a representation for S_{ijmn} which follows directly from (5).

Property 2. *Tensor* $\mathbf{D}$ *is analytic everywhere in* Ω.

It is apparent, from expressions (2), (3), that analyticity of $|\mathbf{x}|$ and $|\mathbf{x}|^{-1}$ implies analyticity of $\mathbf{D}$. The proof for $|\mathbf{x}|$ as well as for $|\mathbf{x}|^{-1}$ becomes elementary if we write $|\mathbf{x}| = \sqrt{\mathbf{x}\cdot\mathbf{x}}$, and observe that $\mathbf{x}\cdot\mathbf{x}$ is analytic everywhere, and the square root function is analytic everywhere except for the origin. As Ω does not contain the origin, the proof is complete.

The next property requires a definition. A Taylor's polynomial of degree n of a smooth function of $\mathbf{x}$ is complete if it omits none of the monomials of degree n.

Property 3. *If* $\mathbf{u}(\mathbf{x})$ *is an analytic solution of the homogeneous Navier equations of elasticity in* Ω, *then its complete Taylor's polynomial is also a homogeneous solution of the Navier equations in* Ω.

This property can be verified by a substitution. It is interesting to observe that this conclusion can be drawn for any linear differential operator with constant coefficients for which the characteristic equation is in the form of homogeneous polynomial. Here, we restrict this property to the Eshelby solution. Let $\bar{\mathbf{D}}$ be a complete Taylor's polynomial of $\mathbf{D}$. Then for a constant strain tensor γ, the strain field $\bar{\mathbf{D}}\gamma$ in the matrix is compatible and the corresponding stress satisfies the equilibrium equations.

Finally, we calculate the volume average of tensor $\mathbf{D}$ inside a spherical subdomain of the matrix. This calculation can be done directly by integration of expressions (2), (3) (Willis and Acton, 1976). We obtain the same result as a consequence of a more general formula, which is an extension of the Gauss related theorem for harmonic functions. A detailed derivation of this formula is given in the Appendix. For a spherical subdomain of the matrix with radius r and the center at $\mathbf{c}(|\mathbf{c}| > r+a)$ we derive

$$\langle \mathbf{D} \rangle = \mathbf{D}(\mathbf{c}) + \frac{a^3 r^2}{30(1-\nu)} \nabla\nabla\nabla\nabla \left(\frac{1}{|\mathbf{x}|} \right)_{|\mathbf{x}=\mathbf{c}}. \tag{6}$$

This integration takes advantage of the fact that potentials $\phi(\mathbf{x})$ and $\psi(\mathbf{x})$ are harmonic and bi-harmonic functions, respectively.

4. THE PROBLEM FOR N INHOMOGENEITIES

Let us now demonstrate how the formalism of Eshelby can be extended, in an approximate manner, to the problem for N non-intersecting spherical inhomogeneities. The inhomogeneities are characterized by stiffnesses $\mathbf{C}^n$ and occupy subdomains ω^n of an infinite elastic body. Remote fields in the matrix, $\boldsymbol{\varepsilon}^{\infty}$ and $\boldsymbol{\sigma}^{\infty}$ $(\boldsymbol{\sigma}^{\infty} = \mathbf{C}\boldsymbol{\varepsilon}^{\infty})$, are uniform. In principle, this problem can be written in terms of $6N$ coupled singular integral equations which are derived as a combination of the equivalency condition (4) and representation (5). For $N > 1$ a closed-form solution to this system of equations does not exist.

A natural approach to the problem for N inhomogeneities is to approximate the equivalent transformation strains by polynomials in Cartesian components (Moschovidis and Mura, 1975). It turns out, however, that this approach may require polynomials of high degree in order to achieve reasonable accuracy (see next section). An attractive treatment of the problem can be based on ideas proposed by Kachanov (1985, 1987) for interacting cracks. Here, we reformulate the principal assumptions of Kachanov in the context of the Eshelby formalism:

- The original problem for N inhomogeneities can be represented by N problems for a single inhomogeneity subject to a remote strain (stress) field induced by the remaining inhomogeneities and the uniform remote strain (stress).
- The contributions of the remaining inhomogeneities to the remote field of a reference inhomogeneity are based only on their average equivalent transformation strains.

The idea of representing the problem for N inhomogeneities as N problems for a single inhomogeneity has been employed in analyses of cracks by several authors (see Kachanov, 1987, for references). We, nevertheless, want to emphasize that its adoption is an assumption which permits us to substitute a complete field by N local fields. This assumption is not ambiguous if we restrict our calculations of the stress field in the matrix to the interfaces only. Furthermore, in view of the second assumption, the most accurate values of interfacial stress concentrations for a given inhomogeneity are obtained from the corresponding local field.

The second assumption implemented in the equivalency condition for the nth inhomogeneity gives

$$\mathbf{C}^n(\boldsymbol{\varepsilon}^\infty + \sum_{m\neq n} \mathbf{D}^m\langle\boldsymbol{\beta}^m\rangle + \boldsymbol{\varepsilon}^n) = \mathbf{C}(\boldsymbol{\varepsilon}^\infty + \sum_{m\neq n} \mathbf{D}^m\langle\boldsymbol{\beta}^m\rangle + \boldsymbol{\varepsilon}^n - \boldsymbol{\beta}^n) \quad \text{in} \quad \omega^n. \tag{7}$$

The second term within each set of parentheses corresponds to the remote strain field induced by the remaining inhomogeneities. Let us take the volume average of (7) in ω^n:

$$\mathbf{C}^n(\boldsymbol{\varepsilon}^\infty + \sum_{m\neq n} \langle\mathbf{D}^{mn}\rangle\langle\boldsymbol{\beta}^m\rangle + \mathbf{S}\langle\boldsymbol{\beta}^n\rangle) = \mathbf{C}(\boldsymbol{\varepsilon}^\infty + \sum_{m\neq n} \langle\mathbf{D}^{mn}\rangle\langle\boldsymbol{\beta}^m\rangle + \mathbf{S}\langle\boldsymbol{\beta}^n\rangle - \langle\boldsymbol{\beta}^n\rangle) \quad \text{in} \quad \omega^n. \tag{8}$$

The average strain inside ω^n, induced by its own equivalent transformation strain, follows from *Property* 1 of the previous section. Tensor $\langle\mathbf{D}^{mn}\rangle$ corresponds to tensor $\mathbf{D}$ of the mth inclusion averaged in ω^n. Evidently, for two inhomogeneities with equal radii $\langle\mathbf{D}^{mn}\rangle = \langle\mathbf{D}^{nm}\rangle$. Equation (8), written for every domain, forms a system of $6N$ linear algebraic equations for the average equivalent transformation strains $\langle\boldsymbol{\beta}^n\rangle$. The matrix of this system is full and, in general, not symmetric. It is, nevertheless, characterized by a dominant main diagonal. After the average equivalent transformation strains have been determined, we can formulate the problems for individual inhomogeneities.

The potential energy release, associated with the presence of inhomogeneities, can be calculated directly in terms of $\langle\boldsymbol{\beta}^n\rangle$ (Mura, 1982, p. 177):

$$\Delta\Pi = -\tfrac{1}{2}\boldsymbol{\sigma}^\infty \cdot \sum_{n=1}^{N} V^n\langle\boldsymbol{\beta}^n\rangle. \tag{9}$$

Further, from (9) we can estimate the overall response of a *finite* block of a composite material of volume V containing spherical inhomogeneities. The simplest way of calculating the overall stiffness is by neglecting interactions between the inhomogeneities and the external surface of the matrix i.e. $\langle\boldsymbol{\beta}^n\rangle$ are calculated as if the inhomogeneities are imbedded in an infinite matrix. In order to obtain an expression for the overall stiffness we introduce a set of tensors $\mathbf{Q}^n$ defined by $\langle\boldsymbol{\beta}^n\rangle = \mathbf{Q}^n\boldsymbol{\sigma}^\infty$. These tensors can be extracted from the inverse matrix of system (8). The expression for the overall stiffness $\langle\mathbf{C}\rangle$ follows immediately from (9)

$$\langle\mathbf{C}\rangle = \left(\mathbf{C}^{-1} + \frac{1}{V}\sum_{n=1}^{N} V^n\mathbf{Q}^n\right)^{-1}. \tag{10}$$

The solution to the problem for inhomogeneity ω^n subjected to the remote strain field $\boldsymbol{\varepsilon}^\infty + \Sigma_{m\neq n}\mathbf{D}^m\langle\boldsymbol{\beta}^m\rangle$ cannot be expressed in elementary functions. There are, nevertheless, a number of ways to construct accurate approximate solutions. An analytical approach, suggested by Moschovidis and Mura (1975), involves a complete Taylor's expansion of tensors $\mathbf{D}^m$ about the center of ω^n with the consequent solution of the problem for remote polynomial fields. This approach is justified by *Properties* 2 and 3 of the previous section. Indeed, if we arrange a Taylor's expansion of $\boldsymbol{\varepsilon}^\infty + \Sigma_{m\neq n}\mathbf{D}^m\langle\boldsymbol{\beta}^m\rangle$ as a sum of collections of constant, linear, quadratic, etc. terms, then (i) this sum converges to the remote strain field (*Property* 2), and (ii) each collection *per se* can be viewed as a remote field (*Property* 3). Thus, for every collection, the necessary and sufficient conditions for the equivalency between the inclusion and the inhomogeneity problems should only be imposed inside ω^n. Another approach to this problem is the finite element method. This option becomes especially attractive because, at this instance, we have to analyze only one simple configuration. Some details of the finite element implementation are discussed in the next section where it is applied to the test problem. In the remainder of the paper we refer to a combination of eqns (8) and the finite element method as a hybrid approach.

5. TEST PROBLEM

As a test for the proposed method we choose the problem for two equal voids. The radius of the voids is equal to a, and the distance between their centers to d. The remote stress is either equal triaxial tension, $\sigma_{ij} = \delta_{ij}$, or uniaxial tension, $\sigma_{ij} = \delta_{i3}\delta_{j3}$. The origin of Cartesian coordinates coincides with the center of the lower (first) void, and the x_3-axis contains both centers (Fig. 1). The approximate solutions to this problem given by Chen and Acrivos (1978a), Moschovidis and Mura (1975), and Willis and Acton (1976) do not provide comprehensive reference data as they lack accuracy for closely-spaced voids. On the other hand, the finite element analysis of this problem is inexpensive and capable of providing satisfactory accuracy. The numerical solution is represented by the hoop stress at point $N(0, 0, a)$, where interactions are probably the strongest, and by the potential energy release per void. These quantities, in terms of micromechanics, characterize concentrations of the microscopic stress and the overall response, respectively.

A typical axisymmetric finite element mesh ($d = 2.1a$) is shown in Fig. 2. It consists of a total of 700 four-node displacement-based elements with 500 elements confined to the ligament. The outside radius is chosen to be equal to $10a$. A simple parametric study confirms acceptable accuracy of this model. The potential energy release is calculated from a modified variational principle (Budiansky *et al.*, 1981) which eliminates convergence problems in evaluations of volume integrals. Computations are performed for the following numerical values $a = 1$, $2.001 \leqslant d \leqslant 3$, $\mu = 0.4$, $\nu = 0.25$. Results of the analysis are given in Table 1 (columns 2 and 4) for the potential energy release, and in Tables 2 and 3 (column 2) for the hoop stress, for the cases of equal triaxial tension and uniaxial tension, respectively. The striking feature of these results is that the potential energy release is very insensitive to the presence of interactions. In uniaxial tension its value changes only by 17% as $2.001 \leqslant d < \infty$. For equal triaxial tension this variation is just 2%.

The implementation of our method to the problem for two voids is as follows.

Step 1. Determine the average equivalent transformation strain from the system of eqns (8):

$$\varepsilon' + \langle \mathbf{D}^{21} \rangle \langle \boldsymbol{\beta}^2 \rangle + \mathbf{S} \langle \boldsymbol{\beta}^1 \rangle - \langle \boldsymbol{\beta}^1 \rangle = 0 \quad \text{in} \quad \omega^1,$$

$$\varepsilon' + \langle \mathbf{D}^{12} \rangle \langle \boldsymbol{\beta}^1 \rangle + \mathbf{S} \langle \boldsymbol{\beta}^2 \rangle - \langle \boldsymbol{\beta}^2 \rangle = 0 \quad \text{in} \quad \omega^2.$$

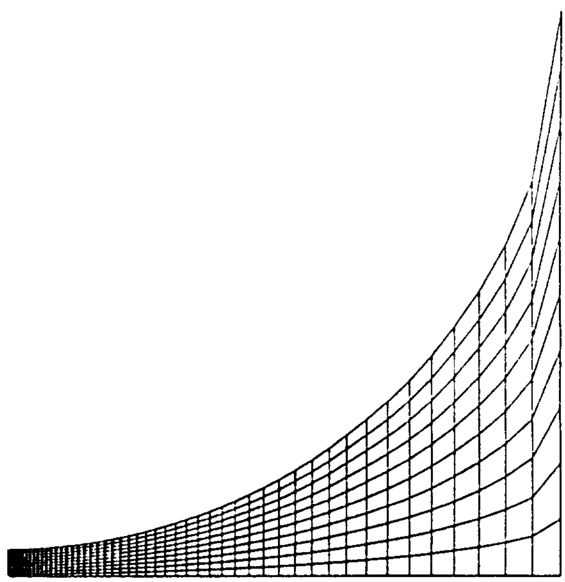

Fig. 2(a). The axisymmetric finite element model of an infinite region containing two spherical voids. The ligament zone.

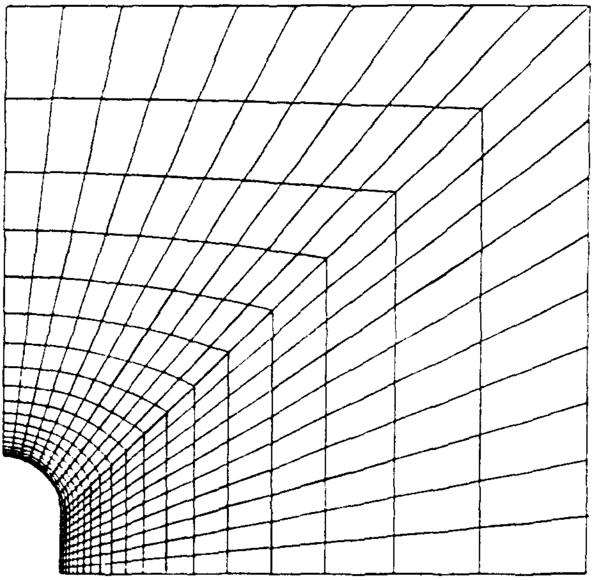

Fig. 2(b). The axisymmetric finite element model of an infinite region containing two spherical voids. The far-field zone.

Tensors $\langle \mathbf{D}^{21} \rangle = \langle \mathbf{D}^{12} \rangle = \langle \mathbf{D} \rangle$ are calculated from (2), (3) and (6). This system of equations is symmetric with respect to $\langle \boldsymbol{\beta}^1 \rangle$ and $\langle \boldsymbol{\beta}^2 \rangle$ therefore we obtain

$$\langle \boldsymbol{\beta}^1 \rangle = \langle \boldsymbol{\beta}^2 \rangle = (\mathbf{I} - \mathbf{S} - \langle \mathbf{D} \rangle)^{-1} \boldsymbol{\varepsilon}^c.$$

Step 2. Calculate the potential energy release associated with the voids (9) (Table 1, columns

Table 1. Potential energy release per inhomogeneity for the remote equal triaxial and uniaxial tension, $a = 1$, $\mu = 0.4$, $\nu = 0.25$

	Triaxial tension		Uniaxial tension	
d	F.E.	Analytical	F.E.	Analytical
2.001	−7.20	−7.17	−3.51	−3.36
2.010	−7.20	−7.17	−3.52	−3.36
2.050	−7.20	−7.16	−3.55	−3.38
2.100	−7.17	−7.15	−3.58	−3.41
2.250	−7.13	−7.12	−3.65	−3.50
2.500	−7.09	−7.10	−3.75	−3.77
3.000	−7.07	−7.08	−3.91	−3.82
∞	−7.07	−7.07	−4.20	−4.20

Table 2. Hoop stress at point N for the remote equal triaxial tension, $a = 1$, $\mu = 0.4$, $\nu = 0.25$

d	F.E.	Analytical	Hybrid
2.001	17.8	2.72	3.74
2.010	7.36	2.68	3.67
2.050	3.99	2.56	3.37
2.100	3.10	2.42	3.08
2.250	2.30	2.15	2.51
2.500	1.92	1.90	2.05
3.000	1.73	1.69	1.73
∞	1.50	1.50	1.50

Table 3. Hoop stress at point N for the remote uniaxial tension, $a = 1$, $\mu = 0.4$, $\nu = 0.25$

d	F.E.	Analytical	Hybrid
2.001	−1.98	−0.213	−0.666
2.010	−0.812	−0.196	−0.601
2.050	−0.426	−0.135	−0.368
2.100	−0.320	−0.089	−0.175
2.250	−0.214	−0.061	0.042
2.500	−0.174	−0.141	−0.009
3.000	−0.319	−0.318	−0.256
∞	−0.589	−0.589	−0.589

3 and 5). Tensors $\mathbf{Q}^r$, for the overall stiffness estimates, are

$$\mathbf{Q}^1 = \mathbf{Q}^2 = (\mathbf{I} - \mathbf{S} - \langle \mathbf{D} \rangle)^{-1}\mathbf{C}^{-1}.$$

Step 3. Solve two individual problems: subject the first void to the remote field $\boldsymbol{\varepsilon}^\infty + \mathbf{D}^2\langle \boldsymbol{\beta}^2 \rangle$, and the second void to $\boldsymbol{\varepsilon}^\infty + \mathbf{D}^1\langle \boldsymbol{\beta}^1 \rangle$. As the two problems are essentially identical, we consider only the first void. The results of the stress calculations using the analytical approach, i.e. approximation of the remote field by the complete Taylor's polynomial of degree two about $\mathbf{x} = \mathbf{0}$, are given in Tables 2 and 3 (column 3). The finite element analysis of this problem follows the basic idea of Eshelby. First, we calculate tractions on the surface of the first void, as if the remote strain was applied to a homogeneous block of material. Second, we apply the opposite tractions in the configuration containing the void so that a superposition of this solution with the remote field $\boldsymbol{\varepsilon}^\infty + \mathbf{D}^2\langle \boldsymbol{\beta}^2 \rangle$ constitutes the traction-free void. Results of the finite element analysis are given in Tables 2 and 3 (column 4). The computations are performed using the axisymmetric mesh shown in Fig. 3. It consists of a total of 2000 eight-node elements. A layer of unit thickness around the void contains 30 equally-spaced elements in a radial direction. Although this discretization is not very cost-efficient, it permits the use of this mesh for all the values of the separation distance d. The angular distribution of the nodes is governed by a geometric progression. Each arc consists of 40 elements.

Table 1 demonstrates that our method provides highly accurate results for the potential energy release associated with the voids. For equal triaxial tension, differences in the predictions between the finite element analysis of this problem and our estimates are of the order of the discretization errors of the finite element model. For uniaxial tension, the

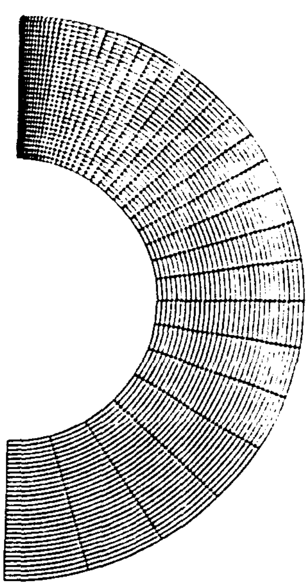

Fig. 3(a). The axisymmetric finite element model of an infinite region containing a spherical void. The near field.

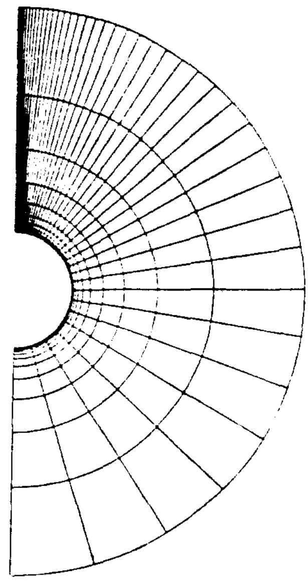

Fig. 3(b). The axisymmetric finite element model of an infinite region containing a spherical void. The far field.

maximum error ($d = 2.001$) is less than 5%. The results of the stress analysis are not as accurate. For equal triaxial tension, the analytical approach is acceptable for $d \geq 2.25$. When $d = 2.25$ the additional stress concentration is $2.15 - 1.50 = 0.65$. The hybrid approach is accurate for $d \geq 2.10$. In this case the additional stress concentration is $3.10 - 1.50 = 1.60$. The correlation is worse in the case of the remote uniaxial tension. Partial explanation for this fact is the apparent non-monotone variation of the hoop stress as the voids approach each other (Table 3, column 2). Although discrepancies in the data may be substantial, it is nevertheless fair to suggest that the hybrid approach gives a reasonable qualitative description of the hoop stress. We also want to mention that the case of uniaxial tension would be the most challenging for the method employed by Chen and Acrivos (1978a).

We conclude this section with an additional test for the analytical approach. The goal of this test is to examine how accuracy of the solution depends upon the degree of the Taylor's polynomials of tensor **D**. Tables 4, 5 and 6 present stress states at point N based on the zeroth-, first- and second-order Taylor's polynomials. These tables correspond respectively to three types of the remote boundary conditions—equal triaxial tension, uniaxial tension, and shear $\sigma_{ij} = \delta_{i2}\delta_{j3} + \delta_{i3}\delta_{j2}$. For the axisymmetric cases we calculate both σ_{11} and σ_{33}, while for the remote shear only σ_{23}. Proximity of stresses σ_{23} and σ_{33} to zero is a good indicator of solutions' quality. The tabulated data clearly demonstrates that

Table 4. The stress state at point N for the remote equal triaxial tension based on the zeroth-, first- and second-order Taylor's polynomials, $a = 1$, $\mu = 0.4$, $\nu = 0.25$

	σ_{11}			σ_{33}		
d	Zeroth order	First order	Second order	Zeroth order	First order	Second order
2.001	2.13	2.35	2.72	−1.10	−0.963	−0.543
2.010	2.11	2.32	2.68	−1.06	−0.932	−0.520
2.050	2.03	2.23	2.56	−0.930	−0.807	−0.432
2.100	1.95	2.13	2.42	−0.796	−0.682	−0.347
2.250	1.80	1.94	2.15	−0.526	−0.436	−0.193
2.500	1.68	1.77	1.90	−0.297	−0.235	−0.086
3.000	1.58	1.63	1.69	−0.120	−0.087	−0.024
∞	1.50	1.50	1.50	−0.000	−0.000	−0.000

Table 5. The stress state at point N for the remote uniaxial tension based on the zeroth-, first- and second-order Taylor's polynomials, $a = 1$, $\mu = 0.4$, $\nu = 0.25$

	σ_{11}			σ_{33}		
d	Zeroth order	First order	Second order	Zeroth order	First order	Second order
2.001	−1.12	−0.773	−0.213	−1.05	−0.722	0.058
2.010	−1.10	−0.748	−0.196	−1.06	−0.731	0.038
2.050	−0.987	−0.656	−0.135	−1.06	−0.752	−0.032
2.100	−0.882	−0.571	−0.089	−1.04	−0.751	−0.086
2.250	−0.698	−0.443	−0.061	−0.894	−0.656	−0.139
2.500	−0.585	−0.399	−0.141	−0.629	−0.461	−0.113
3.000	−0.544	−0.441	−0.318	−0.303	−0.212	−0.048
∞	−0.589	−0.589	−0.589	0.000	0.000	0.000

the second order polynomials do not perform consistently better than the lower order polynomials. Table 6, for example, is characterized by a pattern where, for $d < 3$, the zeroth-order polynomial gives the best results and the first-order polynomial the worst. The anticipated behavior is observed only in Table 4, and as a rule, although it is not shown in the tables, for moderately and well-separated voids ($d \geqslant 3$). Although we appreciate the fact that *Property* 2 does not guarantee fast convergence of the Taylor's expansion, the overall poor performance of the analytical approach is a disappointment. The lengthy algebraic expressions associated with the analytical approach have been implicitly verified by finite element calculations for a single void subjected to polynomial remote stress.

6. DISCUSSION

In this paper we have introduced a method of analysis of the linear elasticity problem for an infinite domain containing a finite number of non-intersecting spherical inhomogeneities. This method is based on the two principal assumptions, stated in Section 4, which permit the formulation of the problem in the context of the Eshelby formalism. Although our presentation is restricted to spherical inhomogeneities, an extension to ellipsoidal inhomogeneities is possible. Indeed, *Property* 1, as it is derived in Section 3, holds for an ellipsoidal inclusion. Therefore, with the same assumptions, the system of eqns (8) can be used to determine the average equivalent transformation strains for ellipsoidal inhomogeneities. In this case, however, tensors $\langle \mathbf{D}^{mn} \rangle$ have to be evaluated numerically which may require a considerable amount of computer time if the number of inhomogeneities is large.

Our method can be employed for estimates of the overall response of composite materials with high volume fractions of inhomogeneities. In this respect, the analysis of the test problem has offered two important observations:

Table 6. The stress state at point N for the remote shear loading based on the zeroth-, first- and second-order Taylor's polynomials, $a = 1$, $\mu = 0.4$, $\nu = 0.25$

	σ_{23}		
d	Zeroth order	First order	Second order
2.001	−1.06	−1.10	−1.07
2.010	−0.995	−1.04	−1.01
2.050	−0.761	−0.806	−0.785
2.100	−0.543	−0.590	−0.574
2.250	−0.183	−0.231	−0.224
2.500	0.004	−0.038	−0.037
3.000	0.042	0.015	0.013
∞	0.000	0.000	0.000

- Our method is capable of very accurate and inexpensive calculations of the potential energy release even when the voids are almost in contact (Table 1).
- The potential energy release, which governs the overall response of composite materials, is almost unaffected by the presence of interactions.

These observations naturally led one to question whether the overall response of composite materials with the second phase in the form of spherical inhomogeneities can be predicted accurately by the dilute solution. In order to examine this proposition we conduct the following numerical experiment.

We take a set of $10 \times 10 \times 10$ cubes composed from rigid spherical particles embedded in an incompressible material with shear modus μ. Each particle's radius is equal to either 0.6 or 1. The rationale for the different size particles is purely computational—a standard random number generator fails to simulate volume fractions above $c = 0.3$ using equal-sized particles. Although the overall shear response of the cube is characterized by three constants $\langle C_{1212}\rangle$, $\langle C_{1313}\rangle$, and $\langle C_{2323}\rangle$, we do not distinguish between the orientations and simply assign three values of the overall shear modulus $\langle\mu\rangle$ per cube. The functional form of the expression for $\langle\mu\rangle$ follows directly from eqn (10)

$$\langle\mu\rangle = \frac{\mu}{1 - \alpha c}, \tag{11}$$

where α has to be determined numerically; $\alpha = 2.5$ in the absence of interactions. The overall response is calculated for twenty-one cubes divided into seven sets. Each set is characterized by a constant volume fraction of the rigid particles. Results of the computations are summarized in Table 7. For each volume fraction we determine nine overall

Table 7. Summary of the numerical simulations with 21 cubes

Volume fraction c	Number of small particles	Number of large particles	Overall shear modulus, $\langle\mu\rangle/\mu$: Computed values			Mean value	Dilute solution	Coefficient of variance %	$([8]-[7])/[7]$ %
[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]
			1.32	1.33	1.36				
0.100	0	24	1.32	1.33	1.36	1.33	1.33	1	0
			1.32	1.33	1.33				
			1.92	1.97	2.05				
0.201	0	48	2.08	2.10	2.03	2.02	2.00	2	−1
			1.97	2.01	2.03				
			4.28	4.40	4.15				
0.302	0	72	4.04	4.23	3.91	4.19	4.00	4	−4
			4.41	4.16	4.10				
			8.71	10.6	7.83				
0.349	108	60	9.27	9.37	8.37	9.27	8.00	9	−14
			9.92	9.09	10.2				
			25.8	27.6	27.6				
0.378	108	67	21.1	44.5	23.5	30.7	20.0	35	−35
			55.5	26.8	24.0				
			417.0	∞	5000.0				
0.395	108	71	294.0	∞	171.0	∞	250	N/A	N/A
			∞	365.0	693.0				
			∞	∞	∞				
0.399	108	72	∞	∞	∞	∞	1450	N/A	N/A
			∞	∞	∞				

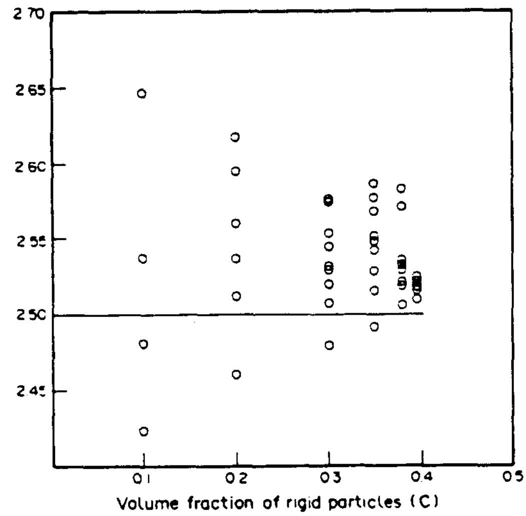

Fig. 4. The data for the parameter α of eqn (11) as obtained from the numerical simulations using 21 cubes.

shear moduli (three per cube), their mean value, and the coefficient of variance. The latter quantity is defined as the ratio of the standard deviation to the mean value. We use the ∞ symbol for the cases when $1-\alpha c \leqslant 0$. An alternative presentation of the results is given as data plot of values of the parameter α for the tabulated values of c (Fig. 4). Apparently, the data can be fitted reasonably well by the line $\alpha = 2.5$. This result incidently coincides with a self-consistent estimate of Hill (1965).

From the conducted numerical experiment, we identify three distinct patterns of behavior. For lower values of the volume fraction of inhomogeneities ($c \leqslant 0.35$) the overall shear modulus does not vary significantly for a fixed value of c. Basically, this fact implies that the volume fraction is an adequate single-parameter representation of the effect of the second phase on the overall response. We also suggest that the overall isotropy of the cubes, in these cases, is better justified by weak interactions among the particles, rather than by randomness of the generated configurations. For $0.35 < c < 0.39$ the difference in the predictions is of the order of the statistical variance of the modulus within the considered configurations of particle. The variance in the values of $\langle \mu \rangle$ can be explained by the proximity of these ensembles of particles to the limit of $c_{max} = 0.4$. This prediction disagrees with the commonly used values of $c_{max} = 0.513$ or $c_{max} = 0.740$ which respectively correspond to cubic and dense packing of uniform spheres. These two limits, however, do not properly represent randomness of particles' positions.† More detailed calculations, which include interactions between the walls of the cube and the particles, may change numerical answers and provide a better explanation.

Formula (11), with $\alpha = 2.5$, agrees well with order c^2 estimate of Chen and Acrivos (1978b)

$$\langle \mu \rangle = \mu[1+2.5c+5.01c^2+\mathcal{O}(c^3)].$$

Our method can be also applied to the analysis of the local stress concentrations in composite materials. The test problem for two voids suggests that the method provides consistently better results if used in combination with the finite element analysis (hybrid approach). Although the need to perform finite element calculations substantially increases the amount of computations it is, nevertheless, reasonable to expect only a small portion of inhomogeneities within a large ensemble will actually require the finite element analysis.

† It is expedient to remark that it is possible to pack uniform spheres at the volume fraction of $c_{max} = 0.123$ (Hilbert and Cohn-Vossen, 1952, p. 51).

The stress state for the majority of inhomogeneities will be practically unperturbed by the neighbors. The test problem demonstrates that our method performs very well qualitatively but predictions of the local stress concentrations may contain inaccuracies. We do not compare our method with those proposed by Moschovidis and Mura (1975) and Willis and Acton (1976). Moschovidis and Mura do not present enough numerical results for such a comparison. Results of Willis and Acton (1976), on the other hand, follow from eqn (8) if the following additional assumptions are made. First, in the evaluation of $\langle \mathbf{D}^{mn} \rangle$ we neglect the second term in the right-hand side of eqn (6). Second, the average equivalent transformation strains $\langle \beta^m \rangle$ of eqn (8) are calculated as if there is only the mth inhomogeneity in the matrix.

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REFERENCES

Batchelor, G. K. and Green, J. T. (1972). The determination of the bulk stress in a suspension of spherical particles to order c^2. *J. Fluid Mech.* **56**, 401–427.

Budiansky, B., Hutchinson, J. W. and Slutsky, S. (1981). Void growth and collapse in viscous solids. In *Mechanics of Solids*, The Rodney Hill 60th Anniversary Volume (Edited by H. G. Hopkins and M. J. Sewell). Pergamon Press, Oxford, pp. 13–45.

Chen, H-S. and Acrivos, A. (1978a). The solution of the equations of linear elasticity for an infinite region containing two spherical inclusions. *Int. J. Solids Structures* **14**, 331–348.

Chen, H-S. and Acrivos, A. (1978b). The effective elastic moduli of composite materials containing spherical inclusions at non-dilute concentrations. *Int. J. Solids Structures* **14**, 349–364.

Christinsen, R. M. (1979). *Mechanics of Composite Materials*. Wiley-Interscience, New York.

Eshelby, J. D. (1957). The determination of the elastic field of an ellipsoidal inclusion and related problems. *Proc. R. Soc. Lond.* Series **A241**, 376–396.

Eshelby, J. D. (1959). The elastic field outside an ellipsoidal inclusion. *Proc. R. Soc. Lond.* Series **A252**, 561–569.

Eshelby, J. D. (1961). Elastic inclusions and inhomogeneities. In *Process in Solid Mechanics* (Edited by I. Sneddon and R. Hill), Vol. 2, pp. 89–140. North-Holland, Amsterdam.

Gradshteyn, I. S. and Ryzhik, I. M. (1980). *Table of Integrals, Series, and Products*. Academic Press, New York.

Hashin, Z. (1983). Analysis of Composite Materials. *J. Appl. Mech.* **50**, 481–505.

Hilbert, D. and Cohn-Vossen, S. (1952). *Geometry and the Imagination*. Chelsea, New York.

Hill, R. (1965). A self-consistent mechanics of composite materials. *J. Mech. Phys. Solids* **13**, 213–222.

Kachanov, M. (1985). A simple technique of stress analysis in elastic solids with many cracks. *Int. J. Fracture* **28**, R11–R19.

Kachanov, M. (1987). Elastic solid with many cracks: a simple method of analysis. *Int. J. Solids Structures* **23**, 23–43.

Moschovidis, Z. A. and Mura, T. (1975). Two ellipsoidal inhomogeneities by the equivalent inclusion method. *J. Appl. Mech.* **42**, 847–852.

Mura, T. (1982). *Micromechanics of Defects in Solids*. Martinus Nijhoff, The Hague–Boston.

Sternberg, E. and Sadowsky, M. A. (1952). On the axisymmetric problem of the theory of elasticity for an infinite region containing two spherical cavities. *J. Appl. Mech.* **19**, 19–27.

Willis, J. R. (1983). The overall elastic response of composite materials. *J. Appl. Mech.* **50**, 1202–1209.

Willis, J. R. and Acton, J. R. (1976). The overall elastic moduli of a dilute suspension of spheres. *Q. J. Mech. Appl. Math.* **29**, 163–177.

APPENDIX: INTEGRATION IN THE DOMAIN OF A SPHERE

The purpose of this appendix is to calculate the volume average of a smooth function prescribed in a sphere. Let a sphere V occupy domain $|\mathbf{x}| \leq a$. First, we evaluate an auxilary volume integral

$$\mathscr{M}_{mnp} = \int_V x_1^m x_2^n x_3^p \, dV, \qquad \text{(A1)}$$

for natural m, n, and p. Elementary calculations using spherical coordinates (Gradshteyn and Ryzhik, 1980, p. 369) give

$$\mathscr{M}_{mnp} = \frac{2a^{m+n+p+3}}{m+n+p+3} \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{m+n+p+3}{2}\right)}, \tag{A2}$$

for even m, n, and p. The integral is equal to zero if any of the numbers is odd. The Γ-function, in this case, can be written in the form

$$\Gamma(k+\tfrac{1}{2}) = \frac{\sqrt{\pi}}{2^k}[1\times 3\times 5\times\cdots\times(2k-1)] = \frac{\sqrt{\pi}}{2^k}(2k-1)!!. \tag{A3}$$

Now we consider the volume average of an arbitrary smooth function $f(\mathbf{x})$

$$\langle f(\mathbf{x})\rangle = \frac{1}{V}\int_V f(\mathbf{x})\,\mathrm{d}V. \tag{A4}$$

The MacLauren series of $f(\mathbf{x})$ is

$$f(\mathbf{x}) = \sum_{N=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty} \frac{1}{m!n!p!}\frac{\partial^N f(0)}{\partial x_1^m\,\partial x_2^n\,\partial x_3^p}x_1^m x_2^n x_3^p. \tag{A5}$$

The summation convention in this formula implies that $m+n+p = N$. If we substitute (A5) into (A4) and use (A2) and (A3) we obtain

$$\langle f(\mathbf{x})\rangle = \sum_{N=0}^{\infty}\frac{3a^{2N}}{2N+3}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{1}{(2m)!(2n)!(2p)!}\frac{\partial^{2N}f(0)}{\partial x_1^{2m}\,\partial x_2^{2n}\,\partial x_3^{2p}}\frac{(2m-1)!!(2n-1)!!(2p-1)!!}{(2N+1)!!}. \tag{A6}$$

We observe that

$$(2k)! = (2k)!!\times(2k-1)!! = 2^k k!\times(2k-1)!!$$

and rewrite (A6) in the form

$$\langle f(\mathbf{x})\rangle = \sum_{N=0}^{\infty}\frac{3a^{2N}}{(2N+3)(2N+1)!}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{N!}{m!n!p!}\frac{\partial^{2N}f(0)}{\partial x_1^{2m}\,\partial x_2^{2n}\,\partial x_3^{2p}}. \tag{A7}$$

Summation over m, n, p is a tri-nomial expansion of Δ^N at $\mathbf{0}$ (Δ is the Laplace operator), and therefore

$$\langle f(\mathbf{x})\rangle = \sum_{N=0}^{\infty}\frac{3a^{2N}}{(2N+3)(2N+1)!}\Delta^N f(\mathbf{0}). \tag{A8}$$

If $f(\mathbf{x})$ is harmonic we recover the well-known result for the average value

$$\langle f(\mathbf{x})\rangle = f(\mathbf{0}), \tag{A9}$$

while for a bi-harmonic function

$$\langle f(\mathbf{x})\rangle = f(\mathbf{0}) + \frac{a^2}{10}\Delta f(\mathbf{0}). \tag{A10}$$